

Notes on static cylindrical shells

J Bičák and M Žofka

Institute of Theoretical Physics, Faculty of Mathematics and Physics, Charles University Prague, V Holešovičkách 2, 180 00 Praha 8, Czech Republic

E-mail: bicak@mbox.troja.mff.cuni.cz, zofka@mbox.troja.mff.cuni.cz

Abstract. Static cylindrical shells made of various types of matter are studied as sources of the vacuum Levi-Civita metrics. Their internal physical properties are related to the two essential parameters of the metrics outside. The total mass per unit length of the cylinders is always less than $\frac{1}{4}$. The results are illustrated by a number of figures.

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1. Introduction

Cylindrically symmetric spacetimes and their sources have belonged to useful models in classical relativity for many years. Although recently attention is paid primarily to dynamical situations in which gravitational waves are present (see, e.g., [1], [2] and references therein) or to the cylindrical fields motivated by string theories (e.g. [3]), unsolved questions persist even in the standard static cases.

In contrast to the Schwarzschild metric described completely by one parameter – the Schwarzschild mass, the static vacuum Levi-Civita (LC) solution contains two essential constant parameters – m , related to the local curvature, and C , determining the global conicity of the spacetime. To connect these two parameters with physical properties of cylindrical sources turns out to be considerably more difficult than in the spherical case. Here we do not wish to survey numerous contributions to the subject – we refer the reader to the recent review [4]; some more references, in particular on static cylindrical shells, are given in the following sections.

The purpose of these notes is to construct various types of physically plausible cylindrical shell sources and to connect their internal properties, especially their mass per unit length, with the two parameters of the external LC metrics. We consider not only the shells made of counter-rotating particles with non-zero rest mass but also of photons. In addition, inspired by recent work on two counter-moving light beams [5], we study shells of counter-spiralling particles with both non-zero and zero rest mass. Next, we discuss perfect fluids as well as shells made of anisotropic matter satisfying, respectively,

the weak, strong, and dominant energy conditions[†]. We determine permissible ranges of radii of the 2-dimensional cylinders made of various types of matter. In all the cases considered, we conclude that the mass per unit length based on the definition by Marder [6] is restricted by $M_1 \leq \frac{1}{4}$. We find that even for small values of the external LC parameter m , it is important to realize that the conicity C enters into the relation between the internal parameter M_1 and the external parameters (solving so a puzzle in literature). Although in most of the text we assume flat spacetime inside the shells, we make some remarks on the situations in which a 'cosmic string' is located along the axis of the cylinder and on the limits that can produce a cosmic string from a 'shrinking' cylinder.

At the end we add a few notes on the geodesics outside the cylinder and, in Appendix, some basic properties of the Newtonian cylindrical shells and the relativistic spherical shells are summarized for comparison. In contrast to the literature on static cylinders we are aware of, we also illustrate the results by a number of figures.

2. Cylindrical shells and their mass per unit length

If the singularity is located on the axis of symmetry then the LC metric in the Weyl coordinates $\tau \in \mathbb{R}, r \in \mathbb{R}^+, \zeta \in \mathbb{R}, \phi \in [0, 2\pi)$ can be written in terms of two constant dimensionless parameters $m \in \mathbb{R}$ and $C \in \mathbb{R}^+$ and constant $r_0 \in \mathbb{R}^+$ with the dimension of length as follows

$$ds^2 = - \left(\frac{r}{r_0} \right)^{2m} d\tau^2 + \left(\frac{r}{r_0} \right)^{2m(m-1)} (d\zeta^2 + dr^2) + r^2 \left(\frac{r}{r_0} \right)^{-2m} \frac{d\phi^2}{C^2}. \quad (1)$$

Introducing dimensionless coordinates $\rho \equiv \frac{r}{r_0}$, $t \equiv \frac{\tau}{r_0}$, $z \equiv \frac{\zeta}{r_0}$, we obtain the metric in the standard dimensionless form (see e.g. [4]):

$$d\tilde{s}^2 = ds^2/r_0^2 = -\rho^{2m} dt^2 + \rho^{-2m} \left[\rho^{2m^2} (dz^2 + d\rho^2) + \frac{1}{C^2} \rho^2 d\phi^2 \right]. \quad (2)$$

In general, we assume two LC solutions to be given in two different sets of these coordinates, one solution inside the shell ($\rho \leq \rho_-$) and the other outside ($\rho \geq \rho_+$). Using Israel's formalism [7] we readily find expressions for the energy-momentum tensor S_{ij} ($i, j = T, Z, \Phi$; the coordinates on the shell are chosen in such a way that the induced 3-metric is flat) induced on the shell:

$$\begin{aligned} 8\pi S_{TT} &= \rho_-^{m_- - m_-^2 - 1} (1 - m_-)^2 - \rho_+^{m_+ - m_+^2 - 1} (1 - m_+)^2, \\ 8\pi S_{ZZ} &= \rho_+^{m_+ - m_+^2 - 1} - \rho_-^{m_- - m_-^2 - 1}, \\ 8\pi S_{\Phi\Phi} &= \rho_+^{m_+ - m_+^2 - 1} m_+^2 - \rho_-^{m_- - m_-^2 - 1} m_-^2, \end{aligned} \quad (3)$$

[†] As demonstrated recently [2], the particular type of the energy condition plays an important role in the outcome of the gravitational collapse of cylindrical shells. (See [2] for a number of references on dynamical cylindrical spacetimes.)

with non-diagonal components vanishing. We require the spacetime to be regular along the axis, thus we put[‡] $m_- = 0$, $C_- = 1$. The proper length of a hoop with t, ρ, z constant must be the same from both sides of the cylinder:

$$2\pi\rho_- = 2\pi\rho_+^{1-m_+}/C_+. \quad (4)$$

We define the mass per unit (coordinate) length of the cylinder, following [6], by

$$M_1 \equiv (\text{Circumference}) \cdot S_{TT} = 2\pi\rho_- S_{TT}, \quad (5)$$

where S_{TT} is given in equation (3). Equations (3) and (5) yield

$$M_1 = \frac{1}{4} \left(1 - \frac{1}{C_+} \frac{(1-m_+)^2}{\rho_+^{m_+^2}} \right). \quad (6)$$

It is immediately seen that $M_1 \leq \frac{1}{4}$. In case of 3-dimensional solid cylinders, an analogous restriction has been recently shown to hold for energy density per unit proper length of the cylinder (see [8], below equation (3.11)). Our result holds for *any* matter on the shell. It is quite in accord with the notion that a spacetime without singularities is free of horizons if the amount of matter within a given region is bounded by a certain finite value. This limiting value can be exceeded only at the expense of a singularity along the axis. Indeed, the case of $M_1 > \frac{1}{4}$ requires $m_- \neq 0$, causing a singularity along the symmetry axis. Admitting $m_- \neq 0$, we find

$$M_1 = \frac{1}{4} \left[(1-m_-^2)(C_+/C_- \rho_+^{1-m_+})^{m_-^2/1-m_-} - (1-m_+)^2 C_-/C_+ \rho_+^{m_+^2} \right],$$

which can attain both positive and negative values. A result similar to (6) can be derived for cylinders in asymptotically anti de Sitter metric. Here again, $M_1 \leq \frac{1}{4}$ if we demand that there be no singularity on the axis, that the cosmological constant outside be greater than inside, and that the outer radius of the shell exceeds its inner radius [9].

By examining expression (6), we find that it is an increasing function of C_+ and ρ_+ . Since the cylindrical shell can be regarded as a source of the outer LC metric one expects the induced linear matter density (6) to increase with the outer "mass parameter" m_+ . However, this is the case only for m_+ in specific intervals depending on the radius ρ_+ of the cylinder – see figure 1. For a finite $\rho_+ > 0$ and sufficiently small m_+ , we do find $\partial M_1 / \partial m_+ > 0$, as expected on classical grounds. Notice, however, that for small ρ_+ , this "intuitive" behaviour does not occur even for small m_+ (the region around the origin in figure 1).

Expanding M_1 into a series for small m_+ we find

$$M_1 \sim \frac{1}{4} \left(1 - \frac{1}{C_+} \right) + \frac{1}{C_+} \frac{m_+}{2} + O(m_+^2). \quad (7)$$

The first term represents the contribution from the angular deficit present in the exterior metric, while the second term results from the influence of the local curvature of

[‡] If we wish to admit a non-zero missing angle (a "cosmic string") inside the shell we need to replace $C_+ \rightarrow C_+/C_-$ in the subsequent formulae. This does not have any major consequences.

do the following replacements: $C_+ \rightarrow C$, $m_+ \rightarrow 2\sigma$, $\rho_+ \rightarrow (Ar_0)^{1/A}$ where

$$A = m_+^2 - m_+ + 1 = 4\sigma^2 - 2\sigma + 1. \quad (8)$$

Now for the shell they consider, we have $C = A^{\frac{1-2\sigma}{A}} r_0^{\frac{-4\sigma^2}{A}} \sim 1 - 2\sigma$ which does not approach 1 fast enough (quadratically in σ) and thus the parameter C influences the limiting value of M_1 . Therefore, the actual value of M_1 must be different from $\sigma = m_+/2$ as it follows from our expansion (7). If in any of these cases we do require $C_+ = 1$, we always obtain the correct Newtonian limit. For Wang et al., this means $r_0 = (A(r_0 - a))^{(1-2\sigma)/A}$ (see their equations (5) and (22), respectively) and then $M_1 = \frac{1}{4}(1 - (1 - 2\sigma)^2 r_0^{\frac{-4\sigma^2}{1-2\sigma}})$, which indeed gives $M_1 \sim \sigma + O(\sigma^2)$. It is essential that for the conicity to be 1 to the second order in σ , one has to have the radial shift a , introduced in references [10], [11], non-vanishing. This is just the case excluded in [10], [11].

Stachel [12] also considered cylindrical shells, however, he was rather interested in the physical meaning of the metric parameters than in the structure of the shell itself. Langer [13], using Israel's formalism, worked out the case of a cylinder composed of two streams of free particles counter-orbiting in $\pm\Phi$ -directions; and he found agreement of his results with those of Raychaudhuri and Som [14] who did not use the Israel formalism but a limiting procedure starting from a shell of a finite thickness. It turns out that in this case the parameter $m_+ \in [0, 1)$. The lower limit is due to the condition of a non-negative matter density whereas the upper one because geodesics become null for $m_+ = 1$. Further, we find $C_+ = \rho_+^{-m_+^2}$ and mass per unit length $M_1 = \frac{1}{4}m_+(2 - m_+)$, independent of other parameters, is always greater than or equal to 0. It has a supremum of $\frac{1}{4}$ at $m_+ = 1$. The formula for M_1 gives the correct weak-field limit. The velocity of the particles as measured by static observers is $v_{(\Phi)} = \sqrt{\frac{m_+}{2-m_+}}$ (see figure 2). If we fix the outer parameters, the inner radius is also fixed at $\rho_- = C_+^{-\frac{A}{m_+^2}}$, with A given in equation (8). By letting C_+ increase from 0 to ∞ , ρ_- decreases from ∞ to 0.

Langer [13] also considered the case of an ideal fluid. Here $M_1 = \frac{1}{2}\frac{m_+}{m_++1}$ is also independent of other parameters and it gives the right Newtonian limit. The admissible range of m_+ is $m_+ \in [0, 1)$; if we require $\mu \geq p$ (μ is the surface energy density), then $m_+ \in [0, \frac{2}{3}]$. The lower limit is due to the non-negativity of energy density, the upper one by requiring a finite pressure. Again, $M_1 \geq 0$, and it has the supremum of $\frac{1}{4}$ for $m_+ = 1$. We have $C_+ = (1 - m_+^2)/\rho_+^{m_+^2}$ and $p \equiv S_{ZZ} = S_{\Phi\Phi} = m_+^2/8\pi\rho_-(1 - m_+^2)$. In figure 3, the mass per unit length and the pressure are illustrated. Using the outer parameters, we can express the inner radius as $\rho_- = C_+^{-\frac{A}{m_+^2}}(1 - m_+^2)^{\frac{1-m_+}{m_+^2}}$, where A is given in equation (8). Increasing C_+ means decreasing ρ_- , as in the case of counter-rotating streams.

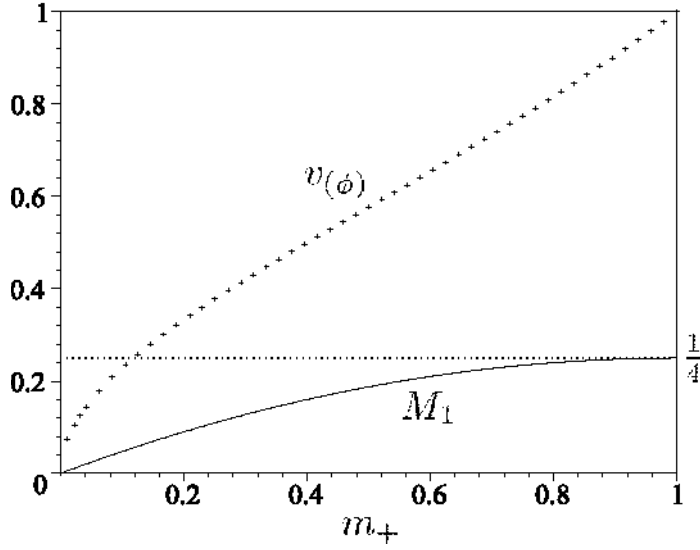


Figure 2. Static cylindrical shells of counter-rotating massive particles: the mass per unit length, M_1 , and the velocity $v_{(\phi)}$ of the particles (measured by static observers) as functions of the Levi-Civita external-field parameter m_+ . With increasing m_+ , both M_1 and $v_{(\phi)}$ increase monotonically up to the limiting values $v_{(\phi)} \rightarrow 1$, $M_1 \rightarrow \frac{1}{4}$.

3. Generalized models of cylindrical shells

The shells from counter-rotating particles in $\pm\Phi$ -directions can be generalized in the following ways:

- (i) We admit counter-streaming particles to move along $\pm Z$ -directions as well ("counter-spiralling motion"). Their velocity can be expressed as $v_{(Z)}^2 \equiv \left(\frac{dZ}{dT}\right)^2 = (1 - \frac{\rho_+^A}{\rho_-}) / (\frac{\rho_+^A}{\rho_-} - (1 - m_+)^2)$ and $v_{(\phi)}^2 = m_+^2 / (\frac{\rho_+^A}{\rho_-} - (1 - m_+)^2)$. We find the mass (6) to read

$$M_1 = \frac{1}{4} \left(1 - \frac{\rho_-}{\rho_+^A} (1 - m_+)^2\right). \quad (9)$$

Necessary conditions for the velocities $v_{(\phi)}, v_{(Z)}$ to be real read $\rho_+^A / \rho_- \in [A, 1]$ and $(m_+ - 1)^2 \leq \rho_+^A / \rho_-$. From here we obtain $m_+ \in [0, 1]$ and $M_1 \in [\frac{m_+}{4A}, \frac{m_+(2-m_+)}{4}]$ so that $M_1 \in [0, \frac{1}{4}]$. Again, $M_1 = \frac{1}{4}$ represents the maximum possible mass per unit length. We can rewrite these conditions in terms of the inner radius as

$$\rho_- \in [C_+^{-\frac{A}{m_+^2}} A^{\frac{1-m_+}{m_+^2}}, C_+^{-\frac{A}{m_+^2}}]. \quad (10)$$

The lower limit for the inner radius corresponds to the case of photons counter-rotating just in $\pm\Phi$ -directions. The upper limit is given by $\mu, p_Z \geq 0$ and $C_+ = \rho_+^{1-m_+} / \rho_-$.

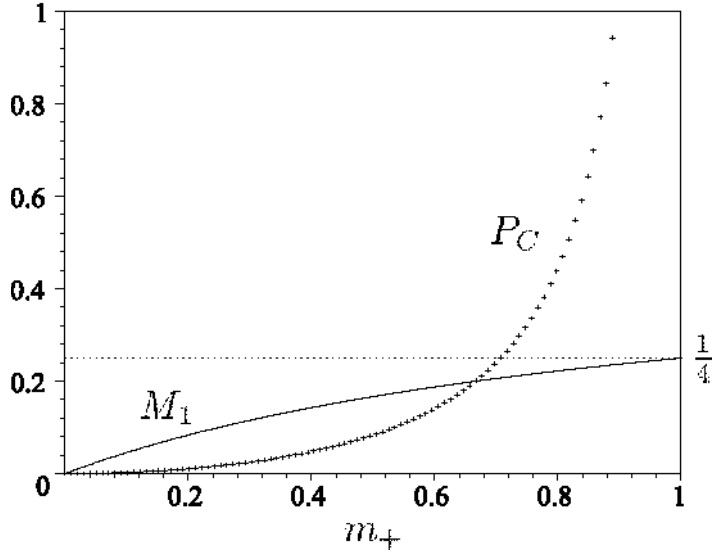


Figure 3. Static cylindrical shells of perfect fluid: the mass per unit length, M_1 , and the magnitude of the surface pressure integrated along a ring, $P_C \equiv (2\pi\rho_-)p$, as functions of the Levi-Civita external-field parameter m_+ . With increasing m_+ , both M_1 and P_C increase monotonically up to the limiting values $P_C \rightarrow +\infty$, $M_1 \rightarrow \frac{1}{4}$.

- (ii) The shell is composed of photons counter-rotating in $\pm\Phi$ -directions (instead of massive particles). We then get $m_+ = 1$ and, consequently, $\rho_- = \rho_+ = 1/C_+$. (In contrast to spherical shells of photons – see Appendix – the cylindrical shells can have arbitrary radii depending on C_+ .) Therefore, $M_1 = \frac{1}{4}$ and the orbiting velocity is equal to 1.
- (iii) We can also consider counter-streaming photons with each stream being in a spiral motion like massive particles in case (1) above. We find $m_+ \in [0, 1]$, $M_1 = \frac{1}{4} \frac{m_+}{A}$. The angular velocity is $v_{(\Phi)}^2 = m_+$ and the velocity along $\pm Z$ -directions reads $v_{(Z)}^2 = 1 - m_+$. The inner radius is at $\rho_- = C_+^{-A/m_+^2} A^{(1-m_+)/m_+^2}$. When $m_+ = 1$ this case reduces to the case (ii). In figure 4, the physical velocities and mass M_1 are plotted as functions of m_+ .

Let us now investigate a general case of 2-dimensional matter with a diagonal stress tensor (see equation (3)) satisfying various types of energy conditions (see, e.g., [15], [16]).

- a) Denoting $S_{TT} = \mu$, $S_{\Phi\Phi} = p_\Phi$, $S_{ZZ} = p_Z$, the weak energy condition, $\mu \geq 0$, $\mu + p_Z \geq 0$, $\mu + p_\Phi \geq 0$, for $m_+ \in [0, 1]$ implies:

$$\rho_- \geq C_+^{-\frac{A}{m_+^2}} (1 - m_+)^{\frac{2(1-m_+)}{m_+^2}}, \quad (11)$$

and for $m_+ \in [1, 2]$ we get

$$\rho_- \leq C_+^{-\frac{A}{m_+^2}} (1 - m_+)^{\frac{2(1-m_+)}{m_+^2}}, \quad (12)$$

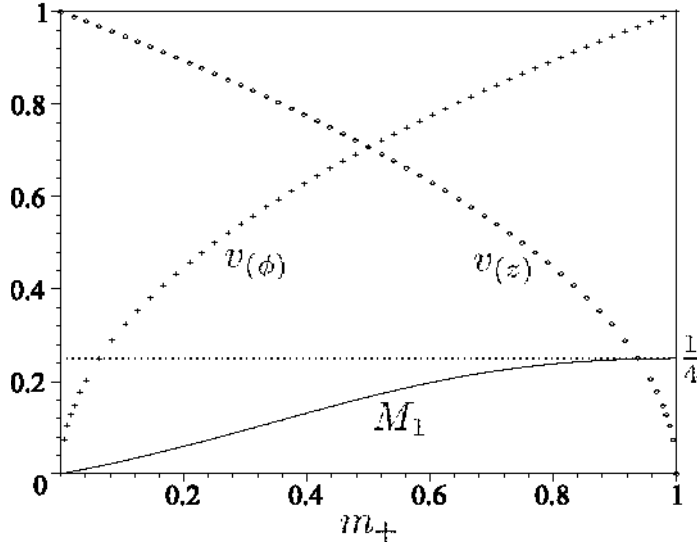


Figure 4. Static cylindrical shells of counter-spiralling photons with physical velocities $v_{(\phi)}$, $v_{(z)}$ ($v_{(\phi)}^2 + v_{(z)}^2 = 1$) as measured by static observers. The mass per unit length, M_1 , and both velocities as functions of the Levi-Civita parameter m_+ are illustrated. With m_+ increasing, M_1 and $v_{(\phi)}$ monotonically increase, whereas $v_{(z)}$ decreases. The limiting point $m_+ = 1$, $v_{(\phi)} = 1$, $v_{(z)} = 0$ corresponds to the point $m_+ = 1$, $v_{(\phi)} = 1$ in figure 2.

the boundary values of ρ_- , corresponding to the equality signs in (11) and (12), are given by the condition $\mu \geq 0$. We find $\rho_+ \geq (\frac{(1-m_+)^2}{C_+})^{\frac{1}{m_+^2}}$, $\frac{\rho_-}{\rho_+^A} \leq \frac{1}{(1-m_+)^2}$ in both intervals of m_+ .

- b) The strong energy condition requires $\mu + p_z + p_\phi \geq 0$, $\mu + p_z \geq 0$, $\mu + p_\phi \geq 0$. For $m_+ \in [0, \frac{1}{2}]$ this implies that

$$\rho_- \geq C_+^{-\frac{A}{m_+^2}} (1 - 2m_+)^{\frac{1-m_+}{m_+^2}}, \quad (13)$$

the boundary value of ρ_- is now given by $\mu + p_\phi \geq 0$. Here we find $\rho_+ \geq (\frac{1-2m_+}{C_+})^{\frac{1}{m_+^2}}$, $\frac{\rho_-}{\rho_+^A} \leq \frac{1}{1-2m_+}$. For $m_+ \in [\frac{1}{2}, 2]$, ρ_- and ρ_+ can be arbitrary.

- c) The dominant energy condition, $\mu \geq |p_z|$, $\mu \geq |p_\phi|$, for $m_+ \in [0, \frac{2}{3}]$ implies that

$$\rho_- \geq C_+^{-\frac{A}{m_+^2}} \left[\frac{m_+^2 - 2m_+ + 2}{2} \right]^{\frac{1-m_+}{m_+^2}}, \quad (14)$$

the boundary value of ρ_- being determined by $\mu \geq p_z$. We find $\rho_+ \geq (\frac{m_+^2 - 2m_+ + 2}{2C_+})^{\frac{1}{m_+^2}}$, $\frac{\rho_-}{\rho_+^A} \leq \frac{1}{1-m_+ + \frac{m_+^2}{2}}$. For $m_+ \in [\frac{2}{3}, 1]$ we get

$$\rho_- \geq C_+^{-\frac{A}{m_+^2}} (2m_+^2 - 2m_+ + 1)^{\frac{1-m_+}{m_+^2}}, \quad (15)$$

and for $m_+ \in [1, 2]$ we finally find

$$\rho_- \leq C_+^{-\frac{A}{m_+^2}} (2m_+^2 - 2m_+ + 1)^{\frac{1-m_+}{m_+^2}}, \quad (16)$$

where the boundary value of ρ_- is given by $\mu \geq p_\phi$. In the last two intervals we get $\rho_+ \geq \left(\frac{2m_+^2 - 2m_+ + 1}{2C_+}\right)^{\frac{1}{m_+^2}}$, $\frac{\rho_-}{\rho_+^A} \leq \frac{1}{1 - 2m_+ + 2m_+^2}$.

Therefore, if m_+ is not large ($m_+ \leq \frac{1}{2}$), there exists a lower limit for possible values of the inner radius ρ_- of the cylinder in case of all three energy conditions. In all three cases we allowed $m_+ \in [0, 2]$ which follows from the condition $\mu + p_Z \geq 0$ (in agreement with Wang et al. [10]). In figures 5 and 6 the above results are illustrated graphically. Let us also note that if we let C_+ change from ∞ to 0, we find ρ_- changing from 0 to ∞ in all three cases.

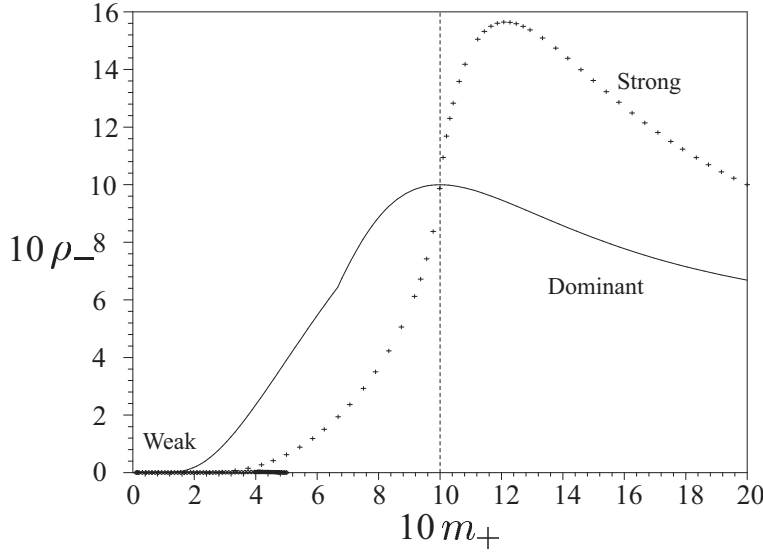


Figure 5. The radii ρ_- of the cylinders satisfying the weak, strong and dominant energy conditions. For $m_+ \in [0, 1]$ the admissible range of radii is *above* the corresponding curve. For $m_+ = 1$, one has $\rho_- = 1/C_+$. (Here, we fixed $C_+ = 1$.) For $m_+ \in (1, 2]$, ρ_- must lay *below* the curves. For the details around the origin, see figure 6.

What are the limiting values of the mass per unit length of the cylinders? We can rewrite the expression for the mass as $M_1 = \frac{1}{4}(1 - \frac{\rho_-}{\rho_+^A}(m_+ - 1)^2)$. From the above inequalities (11) – (16) we obtain $M_1 \in [0, \frac{1}{4})$ for the weak and dominant energy conditions, where the lower limit is reached at the maximum possible value of ρ_-/ρ_+^A , whereas the upper bound results from $\rho_-/\rho_+^A \rightarrow 0$. In case of the strong energy condition and $m_+ \in [0, \frac{1}{2})$ we find that $\frac{1}{4} \geq M_1 \geq -\frac{m_+^2}{4(1-2m_+)}$. However, for $m_+ \in [\frac{1}{2}, 2]$ there is *no* lower bound on M_1 , the upper bound remaining the same. The mass diverges to $-\infty$ as $\rho_+ \rightarrow 0^+$. This can happen since the strong energy condition does not restrict the

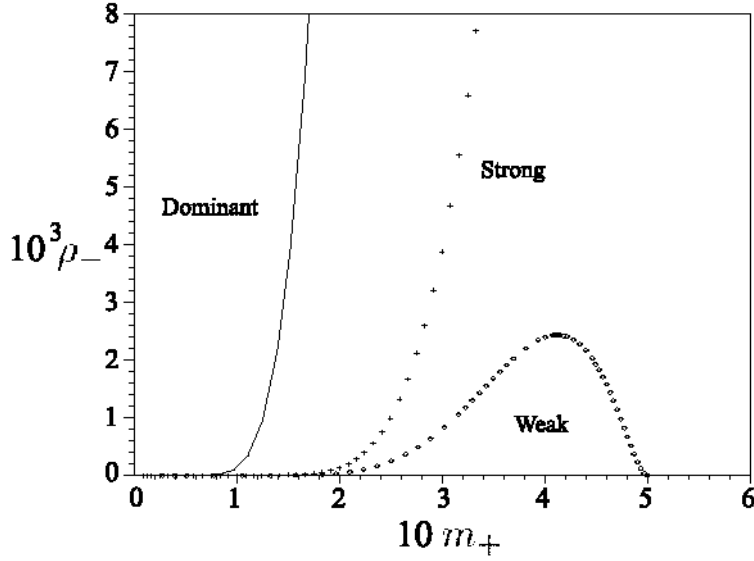


Figure 6. Possible radii ρ_- for small m_+ under various energy conditions (see also the text to figure 5).

value of the density to be non-negative. It is important that in the interval in which it is appropriate to speak about a Newtonian limit ($m_+ \in [0, \frac{1}{2})$) the range of M_1 is still restricted from below.

Summarizing, we see that in the above examples the mass per unit length (equations (6) and (7)) of the cylindrical shell is typically equal neither to $\frac{1}{4}(1 - 1/C_+)$ nor to $m_+/2$ but rather it is a combination of both contributions. As pointed out recently [8], if we keep the parameters of the outer metric fixed while decreasing the outer radius ρ_+ of the shell, we inevitably end up with mass per unit length, M_1 , of the cylinder diverging, $M_1 \rightarrow -\infty$. The limit $\rho_+ \rightarrow 0$ is permitted only under the strong energy condition for values of $m_+ \in [\frac{1}{2}, 2]$, and there is no lower bound on M_1 in this case. Then, however, all other energy conditions are violated except the strong one.

The only two possible candidates for finite-radius models of a cosmic string with finite mass per unit length then are: (i) a shell with $m_+ = 1$, $M_1 = \frac{1}{4}$, and $\rho_- = \frac{1}{C_+}$ which satisfies the weak and strong energy conditions (this is not the case of photons), (ii) a shell with $m_+ = 0$, $C_+ > 1$, $M_1 = \frac{1}{4}(1 - \frac{1}{C_+})$ that satisfies all three energetic conditions, its circumference is equal to zero. Indeed, this is a standard model of a cosmic string [8], [17] – all its effects are caused by a deficit angle in a locally flat spacetime.

In Appendix we present corresponding results for classical cylindrical shells and for relativistic spherical shells.

4. Test particles outside cylinders

Let us look at a radial force exerted upon a free particle at rest in the coordinate system of equation (2). The geodesic equation gives

$$\frac{d^2\rho}{d\tau^2} = -\frac{m}{\rho^{2m^2-2m+1}}. \quad (17)$$

The axis is attractive for $m > 0$, for $m < 0$ it is repulsive. The "magnitude" of the radial acceleration (the absolute value of expression (17)) decreases with increasing ρ (and thus also with the proper distance from the axis as this is an increasing function of ρ) for any non-zero m . However, the behaviour due to the changes in m with ρ kept constant is more subtle (see figure 7). One would expect that a bigger m means a stronger influence of the centre. This, in general, is not the case and, moreover, the behaviour depends on the distance from the axis. For very small distances, $\rho \in (0, e^{-4}]$, and for $m \in (\frac{1}{4}(1 - \sqrt{1 + 4/\ln \rho}), \frac{1}{4}(1 + \sqrt{1 + 4/\ln \rho}))$ the magnitude of the acceleration is decreasing with increasing m . This is against classical intuition since m is positive in the given interval. For $\rho \in (e^{-4}, 1]$ the magnitude is an increasing function of $|m|$ for any m . In the last interval $\rho \in (1, \infty)$ the acceleration behaves intuitively for $m \in (\frac{1}{4}(1 - \sqrt{1 + 4/\ln \rho}), \frac{1}{4}(1 + \sqrt{1 + 4/\ln \rho}))$, while it is counter-intuitive in the remaining range. In [18], the author claims (see p. 1218) that the acceleration decreases with m for $m \in (\frac{1}{2}, 1)$ (increasing magnitude). However, this is not the case for sufficiently large ρ . (Also, he relates the acceleration of particles falling radially from rest to the existence of circular geodesics, which does not appear to be much telling if we recall, for example, the case of the Schwarzschild metric.)

For any value of ρ there exists an interval of m , containing 0, where the acceleration behaves in accordance with classical intuition. The same is true for $\rho > e^{-4}$ and $m \in [0, \frac{1}{2}]$. The first point saves the Newtonian limit, the second provides a classical intuition for this interval of m .

Finally, we study geodesics with $\rho = \text{constant}$. We write the 4-velocity as $U^\alpha = U^t(1, v_z, \omega, 0)$. For photons performing different types of motion we obtain, subsequently

$$\begin{aligned} \text{z-direction} & : v_z = \pm 1, \quad \omega = 0, \quad m = 2 \text{ or } m = 0, \quad U^t, \rho \text{ arbitrary} \\ \varphi\text{-direction} & : v_z = 0, \quad \omega = \pm C, \quad m = \frac{1}{2}, \quad U^t, \rho \text{ arbitrary} \\ \text{spiral motion} & : v_z = \pm \rho^{m(2-m)} \sqrt{\frac{2m-1}{m^2-1}}, \quad \omega = \pm C \rho^{2m-1} \sqrt{\frac{m(m-2)}{m^2-1}}, \\ & m \in < 0, \frac{1}{2} > \cup < 2, \infty), \quad U^t, \rho \text{ arbitrary.} \end{aligned}$$

For particles with non-zero rest mass we find

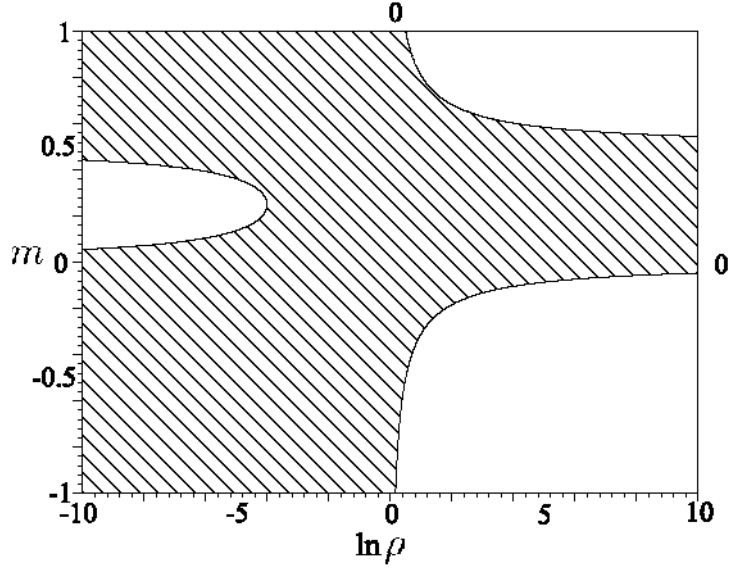


Figure 7. The behaviour of the acceleration of free test particles at fixed radius ρ under the changes of the mass parameter m . In the shaded regions there is $\partial|\frac{d^2\rho}{d\tau^2}|/\partial|m| > 0$.

$$\begin{aligned}
 z\text{-direction} & : v_z = \pm \frac{\rho^{m(2-m)}}{\sqrt{m-1}}, \quad \omega = 0, \quad U^t = \rho^{-m} \sqrt{\frac{m-1}{m-2}}, \quad m > 2, \quad \rho \text{ arbitrary} \\
 & \quad \text{and } v_z = \pm \sqrt{1 - \frac{1}{(U^t)^2}}, \quad m = 0, \quad U^t \geq 1, \quad \rho \text{ arbitrary} \\
 \varphi\text{-direction} & : v_z = 0, \quad \omega = \pm C \rho^{2m-1} \sqrt{\frac{m}{1-m}}, \quad U^t = \rho^{-m} \sqrt{\frac{m-1}{2m-1}}, \\
 & \quad m \in (-\infty, \frac{1}{2}), \quad \rho \text{ arbitrary} \\
 \text{spiral motion} & : v_z = \pm [\rho^{m(1-m)} / U^t] [(1-m + (U^t)^2 \rho^{2m} (2m-1)) / (m^2-1)]^{1/2}, \\
 & \quad \omega = \pm C [\rho^{m-1} / U^t] [m(1-m + (U^t)^2 \rho^{2m} (m-2)) / (m^2-1)]^{1/2}.
 \end{aligned}$$

In the last case of spiral motion, the parameters (m, U^t, ρ) are related by the condition that v_z, ω are real and $v_z^2 + \omega^2 < 1$. There is a solution to these inequalities only as long as $m \in (-\infty, \frac{1}{2}) \cup (2, \infty)$. In other words we can choose two of these parameters and find the admissible range of the remaining parameter. The admissible intervals of m found above in cases of more general than just circular test particle motion correspond to a possible interchange of the roles of coordinates φ and z , suggested in [11].

Appendix

In this Appendix we summarize some basic properties of Newtonian cylindrical shells and of relativistic spherical shells.

A. Newtonian cylindrical shells

Angular velocity of test particles orbiting a cylinder at $\rho = \text{constant}$ is $\omega = \frac{\sqrt{2M_1}}{r}$. The same is true for particles spiralling parallel to the axis. Angular velocity of the particles making up the cylinder of radius R and mass per unit length M_1 is $\omega = \frac{\sqrt{M_1}}{R}$. These particles may rotate in one direction only and move parallel to the axis as well. With a relativistic cylinder we have $\omega = \frac{1}{\rho_-} \sqrt{\frac{1-\sqrt{1-4M_1}}{1+\sqrt{1-4M_1}}}$. If the thin cylinder consists of an ideal fluid, the pressure reads $p = \frac{M_1^2}{2\pi r}$. In relativity we get $p = \frac{M_1^2}{2\pi\rho_-(1-4M_1)}$. In figure (A1) the behaviour of these characteristic functions is illustrated.

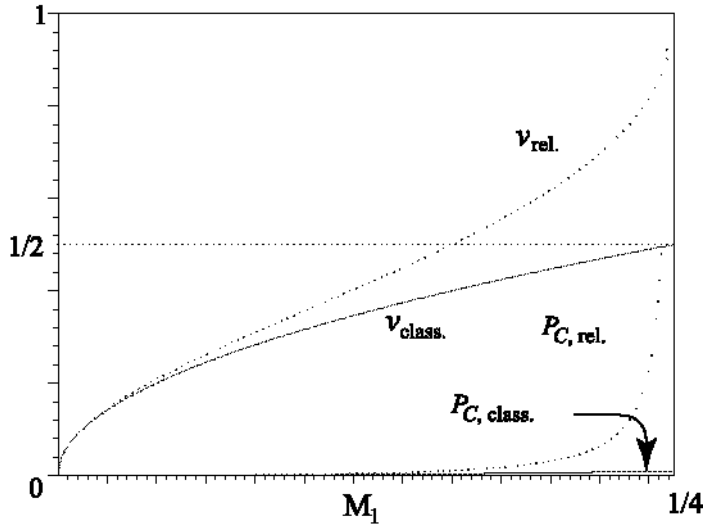


Figure A1. Classical versus relativistic cylindrical shells. We plot $v_{\text{class.}} = r\omega$, $v_{\text{rel.}} = \rho_-\omega$, and 'circumferential' pressures $P_{C, \text{class.}} = (2\pi r)p$ and $P_{C, \text{rel.}} = (2\pi\rho_-)p$ for the physically appropriate range of M_1 .

B. Relativistic thin spherical shells

Let the outer Schwarzschild mass parameter be m . There is a flat spacetime inside. The shell is located at $r_- = r_+ \equiv r$. The mass of the shell is defined as $M \equiv 4\pi r^2 \mu$ with $\mu \equiv S_{TT}$. We get $S_{TT} = \frac{1-\sqrt{F}}{4\pi r}$, $S_{\theta\theta} = \frac{\sqrt{F}-1+\frac{m}{r\sqrt{F}}}{8\pi r}$, and $S_{\phi\phi} = \frac{\sqrt{F}-1+\frac{m}{r\sqrt{F}}}{8\pi r}$, with $ds^2 = -Fdt^2 + \frac{dr^2}{F} + r^2d\theta^2 + r^2\sin^2\theta d\varphi^2$. The principal pressures are the same due to spherical symmetry. We require $r \geq 2m$. For $m \geq 0$ we have $\mu \geq 0$. Regardless of the parameters we find $p \equiv S_{\theta\theta} = S_{\phi\phi} \geq 0$.

Different energy conditions lead to the following results.

- (i) Weak energy condition. One gets $m \geq 0$ if $\mu \geq 0$. This is satisfied for any $r \geq 2m$. We find $M_{\max} = 2m$ for $r = 2m$; however, the pressure then diverges.
- (ii) Strong energy condition. One gets $m \geq 0$ if $\mu + 2p \geq 0$. This is satisfied for any $r \geq 2m$. The case of M_{\max} is the same as in (i).
- (iii) Dominant energy condition. We need $r \geq \frac{25}{12}m$ if $\mu \geq p$. (In this case m may be negative but then also $\mu < 0$ and $p < 0$, with $|p| \geq |\mu|$.) We find $M_{\max} = \frac{5}{3}m$.
- (iv) If the shell is made of photons, we find $r = \frac{9}{4}m$, $M = \frac{3}{2}m$.
- (v) For a shell of particles with non-zero rest mass we obtain $r > \frac{9}{4}m$ while $M_{\max} = \frac{3}{2}m$. For this limiting value of M the trajectories become null.

Maximal M of the shell is again achieved in the case of photons. The dominant energy condition is the most restrictive as in the cylindrical case.

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References

- [1] Bondi H 2000 *Proc. R. Soc. A* **456** 2645
- [2] Goncalves S M C V 2002 *Phys. Rev. D* to appear; see also *Preprint* gr-qc/0202042
- [3] Klepáč P and Horský J 2001 *Czech. J. Phys.* **51** 1178 and gr-qc/0202027
- [4] Bonnor W B 1999 The Static Cylinder in General Relativity in *On Einstein's Path: Essays in Honor of Engelbert Schucking* ed. Harvey A (New York: Springer) 113
- [5] Gönnar U and Kramer D 1999 *Gen. Rel. Grav.* **32** 349
- [6] Marder L 1958 *Proc. R. Soc. A* **244** 524
- [7] Israel W 1966 *Nuovo Cimento* **B 44** 1 (erratum **B 49** 463)
- [8] Anderson M R 1999 *Class. Quantum Grav.* **16** 2845
- [9] Bičák J and Žofka M under preparation
- [10] Wang A Z, da Silva M F A and Santos N O 1997 *Class. Quantum Grav.* **14** 2417
- [11] Herrera L, Santos N O, Teixeira A F F and Wang A Z 2001 *Class. Quantum Grav.* **18** 3847
- [12] Stachel J 1984 *J. Math. Phys.* **25** 338
- [13] Langer J 1969 *Czech. J. Phys.* **19** 807
- [14] Raychaudhuri A K and Som M M 1962 *Proc. Camb. Phil. Soc.* **58** 338
- [15] Hawking S W and Ellis G F R 1973 *The Large-Scale Structure of Space-Time* (Cambridge: Cambridge University Press)
- [16] Wald R 1984 *General Relativity* (Chicago: Chicago University Press)
- [17] Linet B 1986 *J. Math. Phys.* **27** 1817
- [18] Philbin T G 1996 *Class. Quantum Grav.* **13** 1217